International Journal of Theoretical Physics, Vol. 29, No. 12, 1990

Dirac's Constraint Theory: Corrections to Alleged Counterexamples

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Received March 4, 1990

We argue that Dirac's conjecture is valid. Several supposed counterexamples are reexamined in which we do not linearize constraints as do Cawly and other authors.

1. INTRODUCTION

A system with a singular Lagrangian must have Dirac constraints. These constraints are classified into the first and second class. Dirac (1964) proposed that all the first-class constraints are generators of gauge transformations and should be added into the Hamiltonian. Then the extended Hamiltonian must be

$$H_e = H_c + \lambda_m \varphi_m + \mu_l \chi_l \tag{1}$$

H_c is the classical Hamiltonian, and φ_m, χ_l are the primary and secondary first-class constraints respectively.

Dirac's conjecture has been discussed in different ways and on different bases by many authors. Some have agreed with it (Dominici and Gomis, 1980; Appleby, 1982; Castellani, 1982; Gotay, 1983; Di Stefano, 1983; Costa *et al.*, 1985); some have objected to it (Allcock, 1974; Cawly, 1979; Frenkel, 1980; Sugano, 1982; Li, 1989). In this note we argue that Dirac's conjecture is valid.

2. THE VALIDITY OF DIRAC'S CONJECTURE

Suppose a constrained system which was transformed by infinitesimal transformation

$$q_i \rightarrow q_i + \eta_{i(t)}, \qquad p_i \rightarrow p_i + \xi_{i(t)} \tag{2}$$

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has physical invariance. Then its equation of motion (Itzykson and Zuber, 1980)

$$\dot{q}_i = \frac{\partial H_c}{\partial p_i} + \lambda_m \frac{\partial \phi_m}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H_c}{\partial q_i} - \lambda_m \frac{\partial \phi_m}{\partial q_i} \tag{3}$$

and primary constraints $\phi_m \approx 0$ must also be invariant. Paralleling Castellani's (1982) method, we can also obtain

$$\begin{aligned} \{G_0, H_T\} &= \text{primary constraint} \\ G_0 + \{G_1, H_T\} &= \text{primary constraint} \\ G_1 + \{G_2, H_T\} &= \text{primary constraint} \\ &\vdots \\ G_{k-1} + \{G_k, G_T\} &= \text{primary constraint} \\ G_k &= \text{primary constraint} \end{aligned} \tag{4}$$

where k is the number of generators of secondary constraints. All the G_n have to be FC; G_{n-1} is deduced from G_n according to (4); moreover, G_k must be a PFC. These results coincide with Castellani's. The constraints are obtained from the original definition of the canonical momenta or from the consistency condition of the constraints, and we have no reason to reject them.

Based upon this, we have recalculated Cawly's (1979) and other so-called counterexamples. The results verify the validity of Dirac's conjecture.

In addition, the BFV quantum theory (Henneaux, 1985), whose basic equation (Sundermyer, 1982)

$$H = H_0 + v_\alpha T^\alpha + \mu_\alpha \phi^\alpha \tag{5}$$

involves all first-class constraints, supports the validity of Dirac's conjecture.

3. COMMENT ON THE ALLEGED COUNTEREXAMPLES TO DIRAC'S CONJECTURE

In a well-known paper, Cawly (1979) introduced the example

$$L = \dot{X}\dot{Z} + \tfrac{1}{2} Y Z^2 \tag{6}$$

This Lagrangian produces the Euler–Lagrange equation

$$\ddot{Z} = 0, \qquad \tfrac{1}{2} Z^2 = 0, \qquad \ddot{X} = YZ \tag{7}$$

whose solutions are those of uniform X motion, confined to the $X-Y$ plane ($Z = 0$) and with $Y(t)$ remaining undetermined.

There is the primary constraint

$$\phi = P_y = \partial L/\partial \dot{y} \approx 0 \tag{8}$$

and the total Hamiltonian has an arbitrary function v_1:

$$H_T = P_z P_x - \tfrac{1}{2} Y Z^2 + v_1 P_y \tag{9}$$

According to the consistency condition

$$\begin{aligned}
&\dot{\phi} = \{P_y, H_T\} = \tfrac{1}{2} Z^2 \approx 0 \;\Rightarrow\; \chi_1 = Z^2 \approx 0 \\
&\dot{\chi}_1 = \{Z^2, H_T\} = 2ZP_x \approx 0 \;\Rightarrow\; \chi_2 = ZP_x \approx 0 \\
&\dot{\chi}_2 = \{ZP_x, H_T\} = P_x^2 \approx 0 \;\Rightarrow\; \chi_3 = P_x^2 \approx 0 \\
&\dot{\chi}_3 = \{P_x^2, H_T\} = 0
\end{aligned} \tag{10}$$

we obtain the secondary constraints χ_1, χ_2, χ_3.

[But Cawly (1979) linearized the X^n-type constraints, and only obtained two secondary constraints $Z \approx 0$ and $Px \approx 0$.]

All the constraints are first class. So according to Dirac's conjecture, the extended Hamiltonian is

$$H_E = P_z P_x + v_1 P_y + v_2 Z^2 + v_3 Z P_x + v_4 P_x^2 \tag{11}$$

Due to (3), we obtain

$$\dot{X} \approx P_z + v_3 Z + 2 v_4 P_x \tag{12}$$

$$\dot{P}_x \approx 0 \tag{13}$$

$$\dot{Y} \approx v_1 \tag{14}$$

$$\dot{P}_y \approx 0 \tag{15}$$

$$\dot{Z} \approx P_x \tag{16}$$

$$\dot{P}_z \approx -2 v_2 Z - v_3 P_z \tag{17}$$

If $Z \not\approx 0$, using (12), we get

$$Z\dot{X} \approx Z P_x + v_3 Z^2 + 2 y_4 Z P_x$$

Substituting in (10), we obtain

$$\dot{X} \approx P_z$$

If $Z \approx 0$, the result is explicit.

By a similar calculation used in (17), one finds $\dot{P}z \approx 0$. So the equations of motion together with the constraints (8) and (10) correctly express the full content of (7).

Other counterexamples (Allcock, 1975; Gotay, 1983; Di Stefano, 1983) have also been recalculated.

We have not written the constraints in linearized form as do Cawly (1979) and other authors. The equations of motion deriving from H_E are then equivalent to the corresponding Euler–Lagrange equation.

4. CONCLUSION

In Cawly's (1979) and other counterexamples, the authors linearize the X^n-type constraints. Analyzing this method, we find that they confused the concept of weak and strong equality. It is well known that weak and strong equality have the following properties (Dirac, 1958):

$$\begin{aligned} &\text{If } A \simeq 0, \text{then } \delta A \approx 0 \\ &\text{If } A \approx 0, \text{then } \delta A \neq 0 \\ &\text{If } X \approx 0, \text{then } \delta(X^2) = 2X\,\delta X \approx 0 \;\Rightarrow\; X^2 \approx 0 \end{aligned} \tag{18}$$

but we cannot obtain $Z \approx 0$ from $Z^2 \approx 0$, although from $Z \approx 0$ we can find $Z^2 \approx 0$. Thus, the treatment of Cawly (1979) and other authors who linearized the X^n-type constraints is not satisfactory.

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